

On oracle efficiency of the ROAD classification rule

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Abstract

For high-dimensional classification Fishers rule performs poorly due to noise from estimation of the covariance matrix. Fan, Feng and Tong (2012) introduced the ROAD classifier that puts an L_1 -constraint on the classification vector. In their Theorem 1 Fan, Feng and Tong (2012) show that the ROAD classifier asymptotically has the same misclassification rate as the corresponding oracle based classifier. Unfortunately, the proof contains an error. Here we restate the theorem and provide a new proof.

1 Introduction

We consider classification among two groups based on a p -dimensional normally distributed variable. Let the means in the two groups be μ_1 and μ_2 , and let the common variance be Σ . Also, let the probability of belonging to either of the two groups be $\frac{1}{2}$. Defining $\mu_a = (\mu_1 + \mu_2)/2$ and $\mu_d = (\mu_1 - \mu_2)/2$, the Bayes discriminant rule becomes

$$\delta_w(x) = 1 + 1(w^T(x - \mu_a) < 0), \quad \text{with } w = w_F = \Sigma^{-1}\mu_d,$$

where x is classified to group 1 or 2 according to the value of $\delta_w(x)$. The misclassification rate of the rule δ_w is

$$W(\delta_w) = \bar{\Phi}\left(\frac{1}{2}w^T\mu_d/(w^T\Sigma w)^{1/2}\right),$$

where $\bar{\Phi}(z) = 1 - \Phi(z)$ is the upper tail probability of a standard normal distribution. The interpretation of the Bayes rule is that w_F is the vector that minimizes the misclassification rate. Fan, Feng and Tong (2012) suggest to use a L_1 regularized version of w_F , that is,

$$w_c = \arg \min_{\|w\|_1 \leq c, w^T\mu_d=1} w^T\Sigma w.$$

Its sample version

$$\hat{w}_c = \arg \min_{\|w\|_1 \leq c, w^T\hat{\mu}_d=1} w^T\hat{\Sigma}w$$

yields the ROAD classifier

$$\hat{\delta} = 1 + 1(\hat{w}_c^T(x - \hat{\mu}_a) < 0).$$

Theorem 1 of Fan, Feng and Tong (2012) states that the misclassification rate $W(\hat{\delta})$ of the ROAD classifier approaches the misclassification rate of the oracle classifier $W(\delta_{w_c})$. Unfortunately, an essential step in the proof use an inequality which is not valid, see Appendix A for details. We reformulate the theorem and give a new proof.

Theorem 1. *Let ϵ be a positive constant such that $\max_j\{|\mu_{dj}|\} > \epsilon$, and $c > \epsilon + 1/\max_j\{|\mu_{dj}|\}$. Let a_n be a sequence tending to zero such that $\|\hat{\Sigma} - \Sigma\|_\infty = O_p(a_n)$, and $\|\hat{\mu}_i - \mu_i\|_\infty = O_p(a_n)$, $i = 1, 2$. Then, as $n \rightarrow \infty$:*

$$W(\hat{\delta}) - W(\delta_{w_c}) = O_p(d_n)$$

with $d_n = c^2 a_n (1 + c^2 \|\Sigma\|_\infty)$.

Prior to proving the theorem we comment on the differences compared to Theorem 1 of Fan, Feng and Tong (2012). Contrary to us, Fan, Feng and Tong (2012) requires that the smallest eigenvalue of Σ is bounded from below. The upper bound on $W(\hat{\delta}) - W(\delta_{w_c})$ in Fan, Feng and Tong (2012) depends on the sparsity of w_c and of $w_c^{(1)}$, where $w_c^{(1)}$ is given by

$$w_c^{(1)} = \arg \min_{\|w\|_1 \leq c, w^T \hat{\mu}_d = 1} w^T \Sigma w,$$

whereas our bound depends on the regularizing parameter c only. In the formulation of the theorem c is allowed to depend on n . We require a lower bound on $\max_j\{|\mu_{dj}|\}$, which is not part of the theorem in Fan, Feng and Tong (2012). However, it enters indirectly in that we must have $c > 1/\max_j\{|\mu_{dj}|\}$ in order for w_c to exist. Thus, if $\max_j\{|\mu_{dj}|\} \rightarrow 0$, we have $c \rightarrow \infty$, and c enters the upper bound of Fan, Feng and Tong (2012). The reason for our more restrictive condition $c > \epsilon + 1/\max_j\{|\mu_{dj}|\}$ is that the theorem only makes sense if \hat{w}_c exists with probability tending to one. Similarly, whereas Fan, Feng and Tong (2012) have the condition $\|\hat{\mu}_d - \mu_d\|_\infty = O_p(a_n)$, we have $\|\hat{\mu}_i - \mu_i\|_\infty = O_p(a_n)$, $i = 1, 2$, in order to handle a term in the misclassification rate that has been neglected in Fan, Feng and Tong (2012). Finally, $\|\Sigma\|_\infty$ appears in our bound. However, requiring that the variances Σ_{ii} , $i = 1, \dots, p$, are bounded is often encountered in high dimensional settings.

2 Proof of Theorem 1

In the proof we use the following inequalities:

$$|\bar{\Phi}(a(1 + \epsilon)) - \bar{\Phi}(a)| \leq 2\epsilon \text{ for } a > 0 \text{ and } |\epsilon| < 1, \quad (1)$$

$$|\bar{\Phi}((a + \epsilon)^{-1/2}) - \bar{\Phi}(a^{-1/2})| \leq \epsilon \text{ for } a > 0 \text{ and } a + \epsilon > 0. \quad (2)$$

The misclassification rate consists of two terms corresponding to an observation from each of the two groups. The proofs for the two terms are identical, so to simplify we consider the misclassification rate of an observation from group 1 only. Using (1) the misclassification rate of $\hat{\delta}$ becomes

$$\begin{aligned} W(\hat{\delta}) &= \bar{\Phi}\left(\frac{1}{2} \frac{\hat{w}_c^T \hat{\mu}_d + \hat{w}_c^T (\hat{\mu}_1 - \mu_1)}{\sqrt{\hat{w}_c^T \Sigma \hat{w}_c}}\right) = \bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{\hat{w}_c^T \Sigma \hat{w}_c}}\right) + O(|\hat{w}_c^T (\hat{\mu}_1 - \mu_1)|) \\ &\leq \bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{\hat{w}_c^T \Sigma \hat{w}_c}}\right) + O(c \|\hat{\mu}_1 - \mu_1\|_\infty). \end{aligned} \quad (3)$$

Next,

$$|\hat{w}_c^T \Sigma \hat{w}_c - \hat{w}_c^T \hat{\Sigma} \hat{w}_c| \leq c^2 \|\hat{\Sigma} - \Sigma\|_\infty,$$

and from (2) we get

$$\bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{\hat{w}_c^T \Sigma \hat{w}_c}}\right) = \bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{\hat{w}_c^T \hat{\Sigma} \hat{w}_c}}\right) + O(c^2 \|\hat{\Sigma} - \Sigma\|_\infty). \quad (4)$$

From the proof in Fan, Feng and Tong (2012) we see that

$$|\hat{w}_c^T \hat{\Sigma} \hat{w}_c - w_c^{(1)T} \Sigma w_c^{(1)}| \leq c^2 \|\hat{\Sigma} - \Sigma\|_\infty,$$

and thus

$$\bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{\hat{w}_c^T \hat{\Sigma} \hat{w}_c}}\right) = \bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{w_c^{(1)T} \Sigma w_c^{(1)}}}\right) + O(c^2 \|\hat{\Sigma} - \Sigma\|_\infty). \quad (5)$$

Combining (3–5) we have

$$W(\hat{\delta}) = \bar{\Phi}\left(\frac{1}{2} \frac{1}{\sqrt{w_c^{(1)T} \Sigma w_c^{(1)}}}\right) + O(c^2 \|\hat{\Sigma} - \Sigma\|_\infty + c \|\hat{\mu}_1 - \mu_1\|_\infty). \quad (6)$$

Since the oracle misclassification rate is $W(\delta_{w_c}) = \bar{\Phi}(1/(2\sqrt{w_c^T \Sigma w_c}))$ we need to compare $w_c^T \Sigma w_c$ with $w_c^{(1)T} \Sigma w_c^{(1)}$.

To this end let

$$\begin{aligned} A_1 &= \{w : w^T \mu_d = 1, \|w\|_1 \leq c\}, \\ A_2 &= \{w : w^T \hat{\mu}_d = 1, \|w\|_1 \leq c\}. \end{aligned}$$

We want to show that for any $w \in A_1$ there exists $\tilde{w} \in A_2$ such that $w^T \Sigma w$ is close to $\tilde{w}^T \Sigma \tilde{w}$ and vice versa. This means that the minimum of $w^T \Sigma w$ over the set A_1 is close to the minimum over the set A_2 .

Let $w \in A_1$, and define $\tilde{w} = w/(w^T \hat{\mu}_d)$. If $\|\tilde{w}\|_1 \leq c$, we have $\tilde{w} \in A_2$, and

$$w^T \Sigma w = (w^T \hat{\mu}_d)^2 \tilde{w}^T \Sigma \tilde{w} = (1 + O(c \|\hat{\mu}_d - \mu_d\|_\infty))^2 \tilde{w}^T \Sigma \tilde{w}.$$

If instead $\|\tilde{w}\|_1 > c$, we first define $\bar{w} \in A_1$ and then $w^* = \bar{w}/(\bar{w}^T \hat{\mu}_d) \in A_2$. To define \bar{w} assume without loss of generality that $\mu_{d1} = \max_j \{|\mu_{dj}|\}$. Write $w = (w_1, w_{(2)})$ where $w_{(2)}$ is $(p-1)$ -dimensional, and define $\bar{w} = (\bar{w}_1, r w_{(2)})$ with $0 < r < 1$, and \bar{w}_1 chosen such that $\bar{w}^T \mu_d = 1$. The latter requirement implies

$$\bar{w}_1 \mu_{d1} = 1 - r w_{(2)}^T \mu_{d(2)} = 1 - r(1 - w_1 \mu_{d1}).$$

We will show that with $r = 1 - c^2 \|\hat{\mu}_d - \mu_d\|_\infty / (c - 1/\mu_{d1}) = 1 - O(c^2 \|\hat{\mu}_d - \mu_d\|_\infty)$ we have $\|w^*\|_1 \leq c$. From the definition of \bar{w} we have

$$\|\bar{w}\|_1 = |\bar{w}_1| + r \|w_{(2)}\|_1 = \frac{|1 - r(1 - w_1 \mu_{d1})|}{\mu_{d1}} + r(\|w\|_1 - |w_1|).$$

If $1 - r(1 - w_1\mu_{d1}) > 0$ we get

$$\|\bar{w}\|_1 = \frac{1}{\mu_{d1}} + r\left(\|w\|_1 - \frac{1}{\mu_{d1}} + w_1 - |w_1|\right) \leq \frac{1}{\mu_{d1}} + r\left(c - \frac{1}{\mu_{d1}}\right).$$

This shows that $\bar{w} \in A_1$ and $w^* \in A_2$ since

$$\|w^*\|_1 = \frac{\|\bar{w}\|_1}{\bar{w}^T \hat{\mu}_d} \leq \frac{\frac{1}{\mu_{d1}} + r\left(c - \frac{1}{\mu_{d1}}\right)}{1 - c\|\hat{\mu}_d - \mu_d\|_\infty} \leq c,$$

when $r \leq 1 - c^2\|\hat{\mu}_d - \mu_d\|_\infty/(c - 1/\mu_{d1})$. If instead $1 - r(1 - w_1\mu_{d1}) < 0$ we find

$$\|\bar{w}\|_1 = \frac{r-1}{\mu_{d1}} + r\|w\|_1 \leq rc - \frac{1-r}{\mu_{d1}} \leq rc,$$

and $\|w^*\|_1 \leq rc/(1 - c\|\hat{\mu}_d - \mu_d\|_\infty) \leq c$ for $r \leq 1 - c\|\hat{\mu}_d - \mu_d\|_\infty$. The latter condition is satisfied with $r \leq 1 - c^2\|\hat{\mu}_d - \mu_d\|_\infty/(c - 1/\mu_{d1})$. Comparing \bar{w} and w we get

$$\begin{aligned} |w^T \Sigma w - \bar{w}^T \Sigma \bar{w}| &\leq 2c\|w - \bar{w}\|_1 \|\Sigma\|_\infty \leq 2c[(1-r)\|w\|_1 + (1-r)\frac{1}{\mu_{d1}}] \|\Sigma\|_\infty \\ &= O(c^4\|\hat{\mu}_d - \mu_d\|_\infty \|\Sigma\|_\infty), \end{aligned}$$

and also

$$|\bar{w}^T \Sigma \bar{w} - w^{*T} \Sigma w^*| \leq (w^{*T} \Sigma w^*) O(c\|\hat{\mu}_d - \mu_d\|_\infty).$$

We have now shown that any value of $w^T \Sigma w$ for $w \in A_1$ is close to the corresponding value for some $\tilde{w} \in A_2$. The other way around, starting with $w \in A_2$, is treated in the same way. The only difference is that instead of using $c - 1/\mu_{d1} > \epsilon$, we use that when $|\hat{\mu}_{d1} - \mu_{d1}| < \min\{\epsilon, \epsilon^3/(2 + \epsilon^2)\}$, which happens with probability tending to 1 (exponentially fast), we have $\hat{\mu}_{d1} > 0$ and $c - 1/\hat{\mu}_{d1} > \epsilon/2$. Therefore, the minimum $w_c^T \Sigma w_c$ of $w^T \Sigma w$ over the set A_1 is close to the minimum $w_c^{(1)T} \Sigma w_c^{(1)}$ over the set A_2 :

$$\begin{aligned} w_c^T \Sigma w_c &= w_c^{(1)T} \Sigma w_c^{(1)} + O(c^4\|\hat{\mu}_d - \mu_d\|_\infty \|\Sigma\|_\infty) + O(c\|\hat{\mu}_d - \mu_d\|_\infty w_c^T \Sigma w_c) \\ &= w_c^{(1)T} \Sigma w_c^{(1)} + O(c^4\|\hat{\mu}_d - \mu_d\|_\infty \|\Sigma\|_\infty). \end{aligned}$$

Combining the latter with (6) we conclude

$$|W(\hat{\delta}) - W(\delta_{w_c})| = O(c^2 a_n (1 + c^2 \|\Sigma\|_\infty)).$$

Acknowledgement

We thank Xin Tong for reading this note and refer to the arXiv version of Fan, Feng and Tong (2012) for updated versions.

Appendix A

An essential step in the proof in Fan, Feng and Tong (2012) is the inequality (used in equation (21) of that paper)

$$\frac{w_c^T \hat{\mu}_d}{\sqrt{w_c^T \Sigma w_c}} \leq \frac{1}{\sqrt{w_c^{(1)T} \Sigma w_c^{(1)}}}.$$

Unfortunately, this inequality is not correct. We illustrate this by a concrete example. We consider the two-dimensional case with

$$\mu_d = (1, 0)^T, \quad \Sigma = \begin{pmatrix} 1 & 1 \\ 1 & \sigma \end{pmatrix}, \quad c = 1 + \epsilon \text{ with } \epsilon < 1/\sigma.$$

In this case we have

$$w_c = (1, -\epsilon)^T \quad \text{and} \quad w_c^T \Sigma w_c = 1 - 2\epsilon + \sigma\epsilon^2.$$

Consider next $\hat{\mu}_d = (1 + a, b)$ with a and b small. For a and b sufficiently small we obtain

$$w_c^{(1)} = \left(\frac{1 + b[a + \epsilon(1 + a)]/(1 + a + b)}{1 + a}, -\frac{a + \epsilon(1 + a)}{1 + a + b} \right)^T, \quad (7)$$

and

$$w_c^{(1)T} \Sigma w_c^{(1)} = (w_{c1}^{(1)})^2 + 2w_{c1}^{(1)} w_{c2}^{(1)} + \sigma(w_{c2}^{(1)})^2.$$

For a and b small and including $O(a)$ and $O(b)$ terms only we get

$$\frac{1}{\sqrt{w_c^{(1)T} \Sigma w_c^{(1)}}} = \frac{1}{\sqrt{1 - 2\epsilon + \sigma\epsilon^2}} \left\{ 1 + a - b\epsilon + (a - b\epsilon) \frac{1 + \epsilon - \epsilon\sigma(1 + \epsilon)}{1 - 2\epsilon + \sigma\epsilon^2} \right\}, \quad (8)$$

which must be compared to

$$\frac{w_c^T \hat{\mu}_d}{\sqrt{w_c^T \Sigma w_c}} = \frac{1 + a - b\epsilon}{\sqrt{1 - 2\epsilon + \sigma\epsilon^2}}. \quad (9)$$

We thus see that (8) is less than (9) when $a - b\epsilon$ has the opposite sign of $1 + \epsilon - \epsilon\sigma(1 + \epsilon)$. Since $(a - b\epsilon) \sim N(0, (1 - 2\epsilon + \sigma\epsilon)c_0)$ for some constant c_0 , the probability of a particular sign of $a - b\epsilon$ is one half.

References

Fan, J., Feng, Y. and Tong, X. (2012). A road to classification in high dimensional space: the regularized optimal affine discriminant *J. R. Statist. Soc. B*, **74**, 745-771. The paper with revised versions can also be found at arXiv.org: arXiv:1011.6095v2 [stat.ML].